

COHERENT CONTROL OF A QUBIT IS TRAP-FREE

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Abstract

There is a strong interest in optimal manipulating of quantum systems by external controls. Traps are controls which are optimal only locally but not globally. If they exist, they can be serious obstacles to the search of globally optimal controls in numerical and laboratory experiments, and for this reason the analysis of traps attracts considerable attention. In this paper we prove that for a wide range of control problems for two-level quantum systems all locally optimal controls are also globally optimal. Hence we conclude that two-level systems in general are trap-free. In particular, manipulating qubits—two-level quantum systems forming a basic building block for quantum computation—is free of traps for fundamental problems such as the state preparation and gate generation.

1 Introduction

Manipulation of single quantum systems is an important branch of modern science with applications ranging from laser-driven population transfer in atomic systems and laser-assisted control of chemical reactions to quantum technologies and quantum information [1, 2, 3, 4, 5, 6]. The 2012 Nobel Prize in Physics was awarded to Serge Harosche and David Wineland “for groundbreaking experimental methods that enable measuring and manipulation of individual quantum systems” [7].

A fundamental issue is to control qubits, that is, two-state quantum systems which serve as a basic building block for quantum computation and quantum information processing [8, 9, 10, 11, 12, 13, 14, 15, 16]. Physical implementation of qubits includes nuclear spins addressed through nuclear magnetic resonance, electrons in a double quantum dot controlled by small voltages applied to the leads, holes in quantum dots controlled by optical pulses [10], charge states of nanofabricated superconducting electrodes coupled through Josephson junctions, ions in traps [11], polarization or spatial modes of a single photon manipulated using optical elements [12], etc. In any physical implementation, the qubit interacts with the environment, which causes its dynamics to be non-unitary and decreases the performance of control operations. The simplest way to avoid the influence of the environment is to perform fast control operations such that their duration T is significantly smaller than the decoherence time. If this is impossible, a promising method of dynamical decoupling [19] can be used to minimize the influence of the environment. This method has recently been experimentally tested for the Hadamard, NOT, and $U_{\pi/8}$ gates for the gate time T exceeding the decoherence time by the order of magnitude [20].

Any physical implementation of the qubit requires the ability to optimally prepare in a controlled manner arbitrary superpositions of the two qubit basis states and produce arbitrary

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single-qubit quantum circuits. Finding controls which optimally achieve these goals is crucial for laboratory implementation of various quantum computing schemes [14]. Often the search for optimal controls is performed using numerical methods (see, e.g., [17, 18]) including the gradient methods (see [21]).

Traps are controls which are optimal only locally but not globally. Arbitrary small variations of a trapping control do not increase the performance of the target (e.g., a circuit operation), but globally their outcomes can be far from good. Locally, traps look optimal, and if they exist, they can be serious obstacles to finding desired globally optimal controls and can significantly slow down or even completely prevent finding such solutions in numerical and laboratory experiments. For this reason the analysis of traps has recently attracted much attention [22, 23, 24, 25, 26, 27, 28, 29, 30]. Despite of these extensive studies, the absence of traps has been proved only for the two-level Landau–Zener system [28] and for the control of the transmission coefficient of a quantum particle passing through a potential barrier [31]. Moreover, trapping behavior has been revealed for three-level and multi-level quantum systems [25, 29].

The present paper contributes significantly to the field by showing that the control of general two-level systems is completely free of traps for many fundamental problems including those of optimal state preparation and single qubit gate generation.

In this paper we assume that the environmental influence can be avoided so that the Schrödinger equation provides a reasonable approximation for the qubit evolution. We assume that the system is controllable so that available controls are sufficient to produce any unitary evolution. As was shown numerically and theoretically for the Landau–Zener system, these assumptions can be significantly relaxed while still keeping the trap-free behavior [28]. We also consider manipulating a single qubit. Important problems involving control of multi-qubit dynamics, as necessary, for example, for producing entangled states or a C-NOT gate, are beyond the scope of this work.

2 Formulation

We consider coherent control of a two-level quantum system which evolves under the action of coherent control $f(t) \in \mathcal{U} = L^1([0, T]; \mathbb{R})$ ($T > 0$ is some final time) according to the Schrödinger equation

$$i \frac{dU_t^f}{dt} = (H_0 + V f(t)) U_t^f, \quad U_{t=0}^f = \mathbb{I}$$

Here free and interaction Hamiltonians $H_0, V \in \mathbb{C}^{2 \times 2}$ are two-by-two Hermitian matrices. Evolution is unitary, $U_t^f \in U(2)$. The components of the matrix U_t^f belong to the space of absolutely continuous functions on the interval $[0, T]$, $U_t^f \in AC[0, T]$.

Many important quantum control problems are terminal-time control problems, where the goal is to maximize an objective at a specific final time T . Such objectives have the form

$$\mathcal{F}(f) = \mathcal{F}(U_T^f)$$

where $\mathcal{F} : U(2) \rightarrow \mathbb{R}$ is a function on the unitary group. For definiteness, we consider maximization of the objective as the control goal, $\mathcal{F}(f) \rightarrow \max$. The function \mathcal{F} is assumed to be phase invariant, that is $\mathcal{F}(U e^{i\phi}) = \mathcal{F}(U)$ for any ϕ , to reflect physical equivalence of states which differ only by a phase factor. Thus without loss of generality, we can naturally identify any $U_T^f \in U(2)$ with an element of $SU(2)$ and introduce the map $\Phi : \mathcal{U} \rightarrow SU(2)$ defined as $\Phi(f) = U_T^f / \sqrt{\det U_T^f}$. It is important to emphasize that the objective is a functional of the control whereas \mathcal{F} is a function of a unitary matrix.

The graph of the objective functional $\mathcal{F}(f)$ is the *dynamic control landscape*. The graph of the function $\mathcal{F}(U)$ is the *kinematic control landscape*. Control f is a *trap* if f is a local but

not a global maximum of \mathcal{F} . Control f is a *second-order trap* if f is a critical point, that is $\delta\mathcal{F}/\delta f = 0$, Hessian of \mathcal{F} at f is negative semidefinite, that is $\delta^2\mathcal{F}/\delta f^2 \leq 0$, and f is not a global maximum of \mathcal{F} . Control f is *regular* (or *non-degenerate*) if the differential $D_f\Phi$ of the map has maximal rank. The goal of the analysis of the control landscape is to find all traps of the objective functional $\mathcal{F}(f)$ or to prove that there are no traps.

Among major requirements for any implementation of the qubit are the ability to optimally prepare arbitrary qubit states and produce arbitrary unitary evolutions representing single-qubit quantum gates. To achieve these goals, one has to act on the qubit with an external control $f(t)$, e.g. shaped laser pulse, small voltage, etc., which maximizes a desired objective outcome. The objective for steering the system from the initial state $|i\rangle$ into a desired final state $|f\rangle$ at time T is the transition probability

$$\mathcal{F}_{i \rightarrow f}(f) = P_{i \rightarrow f} = |\langle f | U_T^f | i \rangle|^2$$

where U_T^f is the evolution operator of the system at time T induced by the control f . This objective is maximized by any control $f(t)$ such that $U_T^f |i\rangle = e^{i\varphi} |f\rangle$, where φ is an arbitrary (generally physically meaningless) phase. The corresponding objective maximum is $\max_f \mathcal{F}_{i \rightarrow f}(f) = 1$.

The transition probability $\mathcal{F}_{i \rightarrow f}(f)$ is a particular kind of objectives of the form

$$\mathcal{F}_O = \langle O \rangle_T = \text{Tr}[U_T^f \rho_0 U_T^{f\dagger} O] = \text{Tr}[\rho_T^f O],$$

where ρ_0 is the initial system density matrix and O is a Hermitian operator. Such objectives describe the problem of maximizing the average value of the system observable O at time T . The transition probability $\mathcal{F}_{i \rightarrow f}(f)$ corresponds to $\rho_0 = |i\rangle\langle i|$ and $O = |f\rangle\langle f|$. The analysis of traps for \mathcal{F}_O for a two-level system is equivalent to the case when O is a projector. Indeed, for a two-level system any O has a representation $O = \lambda_1 P_1 + \lambda_2 P_2$, where P_1 and P_2 are two orthogonal projectors such that $P_1 + P_2 = \mathbb{I}$, and λ_1 and λ_2 are two eigenvalues. Thus $\text{Tr}[\rho_T^f O] = \lambda_1 + (\lambda_2 - \lambda_1) \text{Tr}[\rho_T^f P_2]$, and in the non-degenerate case ($\lambda_1 \neq \lambda_2$) all traps of $\text{Tr}[\rho_T^f O]$ coincide with traps of $\text{Tr}[\rho_T^f P_2]$. The degenerate case $\lambda_1 = \lambda_2$ is trivial since in this case the objective takes the constant value $\mathcal{F}(f) = \lambda_1$ and traps do not exist. Therefore without loss of generality we can consider O as a projector, $O = |f\rangle\langle f|$. We denote by ω_0 and ω_1 two eigenvalues of ρ_0 and consider non-degenerate case $\omega_0 \neq \omega_1$ since the degenerate case $\omega_0 = \omega_1 = 0.5$ is trivial as producing a constant objective value $\mathcal{F}(f) = \text{Tr} O$.

The objective for generating a desired unitary gate W is

$$\mathcal{F}_W(f) = \frac{1}{4} |\text{Tr}(W^\dagger U_T^f)|^2.$$

Examples for W include Hadamard gate $W = \mathbb{H}$, phase shift gate $W = U_\phi$, etc. This objective is maximized by any $U_T^f = e^{i\varphi} W$, where φ is an arbitrary phase. The normalization factor $1/4$ is chosen to have $\max_f \mathcal{F}_W = 1$ and the absolute value is used to exclude the physically meaningless overall phase of the unitary operator.

We consider arbitrary H_0 and V assuming only that $[H_0, V] \neq 0$ to have non-trivial quantum control properties. In this case $[H_0, V]$ and V are linearly independent. Indeed, assume $\alpha V + \beta[H_0, V] = 0$ for some α, β such that $|\alpha| + |\beta| > 0$. Multiplying this equality by V either from the left or from the right and taking trace gives $\alpha \text{Tr}(V^2) = 0$, that implies $\alpha = 0$, since V^2 for a Hermitian V is positive. Then $\beta[H_0, V] = 0$ which for $[H_0, V] \neq 0$ implies $\beta = 0$. This contradicts the assumption $|\alpha| + |\beta| > 0$ and therefore $[H_0, V]$ and V can not be linearly dependent.

3 Main result

Let $\mathcal{M}_2 := \text{Mat}(2, \mathbb{C})$ be the complex vector space of 2×2 matrices. Denote $f_0 := -[\text{Tr} V \text{Tr} H_0 + 2\text{Tr}(H_0 V)]/[(\text{Tr} V)^2 + 2\text{Tr}(V^2)]$. The key result for our analysis is the following lemma.

Lemma 1 *Let $V_t = U_t^{\dagger} V U_t^f$ and suppose that the function f is not equal to the constant function f_0 . Under this assumption if a linear map $L : \mathcal{M}_2 \rightarrow \mathbb{R}$ satisfies $L(\mathbb{I}) = L(V_t) = 0$ for all $t \in [0, T]$, then $L \equiv 0$.*

Proof. To prove the lemma, consider the function $l(t) := L(V_t)$. The equality $L(V_t) = 0$ means $l(t) \equiv 0$. Therefore, in particular, $l(t) = l'(t) = l''(t) = 0$, that implies

$$0 = L(U_t^{\dagger} V U_t) \quad (1)$$

$$0 = L(U_t^{\dagger} [H_0, V] U_t) \quad (2)$$

$$0 = L(U_t^{\dagger} ([H_0, [H_0, V]] + f(t)[V, [H_0, V]]) U_t) \quad (3)$$

We now show that if function f is not equal to the function f_0 then there exists t such that the matrices \mathbb{I} , V , $[H_0, V]$, and $E_t = [H_0, [H_0, V]] + f(t)[V, [H_0, V]]$ are linearly independent. Indeed, suppose that for all t

$$\alpha_t \mathbb{I} + \beta_t V + \gamma_t [H_0, V] + \delta_t E_t = 0 \quad (4)$$

where complex numbers α_t , β_t , γ_t , and δ_t satisfy

$$|\alpha_t| + |\beta_t| + |\gamma_t| + |\delta_t| > 0 \quad (5)$$

Multiplying this equality either by V or by H_0 from the left and taking trace, together with simply taking trace of Eq. (4), gives the system of equations

$$\begin{aligned} 0 &= \alpha_t \text{Tr} V + \beta_t \text{Tr} V^2 - \delta_t \text{Tr}([H_0, V])^2 \\ 0 &= \alpha_t \text{Tr} H + \beta_t \text{Tr}(H_0 V) + \delta_t f(t) \text{Tr}([H_0, V])^2 \\ 0 &= 2\alpha_t + \beta_t \text{Tr} V \end{aligned}$$

This system is compatible only if $f(t) = f_0$ (recall that $[H_0, V]$ is anti-Hermitian and $[H_0, V] \neq 0$; hence $\text{Tr}([H_0, V])^2 \neq 0$). If $f(t) \neq f_0$ for some t , then this system has only a trivial solution and the assumption of linear dependence (4) with the requirement (5) leads to contradiction. Therefore for any t such that $f(t) \neq f_0$ the matrices \mathbb{I} , V , $[H_0, V]$ and E_t are linearly independent 2×2 matrices. Their unitary evolutions \mathbb{I} , $U_t^{\dagger} V U_t$, $U_t^{\dagger} [H_0, V] U_t$ and $U_t^{\dagger} E_t U_t$ are also linearly independent 2×2 matrices. They form a basis of \mathcal{M}_2 and hence the equations (1)–(3) together with the assumption $L(\mathbb{I}) = 0$ imply that $L(A) = 0$ for any $A \in \mathcal{M}_2$. This proves the lemma.

Remark 1 *The exceptional control value f_0 in the common case of traceless interaction $\text{Tr} V = 0$ takes a simpler form $f_0 = -\text{Tr}(H_0 V)/\text{Tr}(V^2)$. If, in addition, all diagonal elements of V are zero in the basis of the free Hamiltonian H_0 (the most common case), then $f_0 = 0$.*

The exceptional control f_0 is not a trap if T is sufficiently large and $\text{Tr} V = 0$, as stated in the following lemma. Note that time should be large enough also to ensure controllability of the system.

Lemma 2 *Let $\text{Tr} V = 0$ and $T \geq \pi/(\|H_0 - \frac{1}{2}\text{Tr} H_0 + f_0 V\|)$, where $\|\cdot\|$ is the matrix spectral norm. If $\mathcal{F}(U)$ has no traps on $U(2)$, then the control $f(t) = f_0$ is not a trap for $\mathcal{F}(f) := \mathcal{F}(U_T^f)$.*

Proof. The evolution of the system under the action of the control $f(t) = f_0 + \delta f(t)$, where δf is a small variation, is governed by the Schrödinger equation

$$i\dot{U}_t^{\delta f} = (H'_0 + \delta f(t)V)U_t^{\delta f} \quad (6)$$

where $H'_0 = H_0 + f_0V$. The modified free Hamiltonian can be written as $H'_0 = \frac{1}{2}\text{Tr}(H'_0)\mathbb{I} + H''_0$, where H''_0 is traceless. The first term is proportional to the identity matrix and can be neglected. The second term in the suitable basis can be written as $H''_0 = \omega_0\sigma_z$, $\omega_0 > 0$ and by suitably rescaling time we can set $\omega_0 = 1$. Thus, instead of the evolution equation (6) we can consider the equivalent equation

$$i\dot{U}_t^{\delta f} = (\sigma_z + \delta f(t)V)U_t^{\delta f} \quad (7)$$

Checking if f_0 is not a trap for eq. (6) is equivalent to checking if $f(t) = 0$ is not a trap for eq. (7).

The interaction can be written as $V = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z + v_0\mathbb{I}$. We consider the non-trivial case $v = \sqrt{v_x^2 + v_y^2} \neq 0$. The evolution operator produced by $\delta f(t) = 0$ has the form $U_t^0 = e^{-it\sigma_z}$. Introducing the angle $\phi = \arctan(v_y/v_x)$, we can write $V_t := V_t^0 := U_t^{0\dagger}VU_t^0 = v\cos(2t - \phi)\sigma_x - v\sin(2t - \phi)\sigma_y + v_z\sigma_z + v_0$. This gives for the gradient of the objective

$$\nabla \mathcal{F}_{f_0}(t) = v\cos(2t - \phi)L(\sigma_x) - v\sin(2t - \phi)L(\sigma_y) + v_zL(\sigma_z)$$

Suppose $v_z \neq 0$ or $L(\sigma_z) = 0$. If f_0 is a critical point, then the gradient $\nabla \mathcal{F}_{f_0}(t) = 0$ for any $t \in [0, T]$ and, therefore $L(\sigma_x) = L(\sigma_y) = 0$. In addition, $L(\mathbb{I}) = 0$ for any phase-invariant objective and hence $L \equiv 0$ on \mathcal{M}_2 . Then similarly to the proof of the Theorem 1 we conclude that $f = f_0$ is not a trap (it can be either a global maximum or a global minimum).

Now consider the case $v_z = 0$ and $L(\sigma_z) \neq 0$. For this case we assume in addition that the interaction is traceless, that is $v_0 = \text{Tr}V = 0$. The evolution operator produced by a small variation of the control δf can be represented as $U_T^{\delta f} = U_T^0\tilde{U}_T$, where $U_T^0 = e^{-iT\sigma_z}$ and \tilde{U}_T satisfies

$$\dot{\tilde{U}}_t^{\delta f} = -i\delta f(t)V_t^0\tilde{U}_t^{\delta f}, \quad \tilde{U}_0^{\delta f} = \mathbb{I}$$

The operator \tilde{U}_T can be computed up to the second order in δf as

$$\begin{aligned} \tilde{U}_T^{\delta f} &= \mathbb{I} + A_1 + A_2 + o(\|\delta f\|^2) \\ A_1 &= -i \int_0^T dt \delta f(t) V_t^0, \\ A_2 &= - \int_0^T dt_1 \int_0^{t_1} dt_2 \delta f(t_1) \delta f(t_2) V_{t_1}^0 V_{t_2}^0 \end{aligned}$$

We choose δf_1 and δf_2 such that $A_1 = 0$, that is,

$$\int_0^T dt \delta f_k(t) \cos 2t = \int_0^T dt \delta f_k(t) \sin 2t = 0 \quad (k = 1, 2) \quad (8)$$

For such δf_k noting that $V_{t_1}^0 V_{t_2}^0 = v^2[\cos 2(t_1 - t_2) + i\sigma_z \sin 2(t_1 - t_2)]$, we get $A_2 = -iI(\delta f_k)\sigma_z$, where

$$I(\delta f) = v^2 \int_0^T dt_1 \int_0^{t_1} dt_2 \delta f(t_1) \delta f(t_2) \sin 2(t_1 - t_2).$$

Then, up to the second order in δf we have

$$\begin{aligned} \mathcal{F}(\delta f) &= \mathcal{F}(U_T^0(\mathbb{I} + A_2 + \dots)) \\ &= \mathcal{F}(U_T^0) + \text{Tr} \left(\frac{\delta \mathcal{F}}{\delta U} \Big|_{U=U_T^0} U_T^0 A_2 \right) + o(\|\delta f\|^2) \\ &= \mathcal{F}(U_T^0) + I(\delta f)L(\sigma_z) + o(\|\delta f\|^2) \end{aligned}$$

Now suppose $T \geq \pi$ in the rescaled time frame (that corresponds to $T \geq \pi/(\|H_0 - \frac{1}{2}\text{Tr}H_0 + f_0V\|)$ in the original time frame). We will show the existence of variations δf_1 and δf_2 which satisfy Eq. (8) and produce $I(\delta f_1)$ and $I(\delta f_2)$ with opposite signs. An example is $\delta f_1(t) = \chi_{[0,\pi]}(t)$ and $\delta f_2(t) = \cos(4t)\chi_{[0,\pi]}(t)$, where $\chi_{[0,\pi]}(t)$ is the characteristic function of the interval $[0, \pi]$. For these variations $I(\delta f_1) = \pi v^2/2$ and $I(\delta f_2) = -\pi v^2/12$. Therefore for $L(\sigma_z) \neq 0$ there exist directions at $f(t) = 0$ in which the objective increases and directions in which it decreases. This means that $f(t) = 0$ for Eq. (7) (and thus $f(t) = f_0$ for Eq. (6)) is neither a local maximum nor a local minimum, and hence is not a trap. This proves the lemma.

Our main result is the following theorem.

Theorem 1 *Suppose the only extrema of the kinematic landscape $\mathcal{F}(U)$ are global maxima and global minima. If $\text{Tr}V = 0$ and $T \geq \pi/(\|H_0 - \frac{1}{2}\text{Tr}H_0 + f_0V\|)$, then the only extrema of the dynamic landscape $\mathcal{F}(f)$ are global maxima and global minima.*

Proof. Consider first the case $f(t) \neq f_0$. The variation of U_T^f has the form $\delta U_T^f / \delta f(t) = -iU_T^f V_t^f$, where $V_t^f = U_t^{f\dagger} V U_t^f$. By the chain rule,

$$\frac{\delta \mathcal{F}}{\delta f(t)} = \text{Tr} \left(\frac{\delta \mathcal{F}}{\delta U} \bigg|_{U=U_T^f} \frac{\delta U_T^f}{\delta f(t)} \right) = -i \text{Tr} \left(\frac{\delta \mathcal{F}}{\delta U} U_T^f V_t^f \right) =: L(V_t^f)$$

Denoting $X = -i(\delta \mathcal{F} / \delta U) U_T$, we get $L(A) = \text{Tr}(XA)$. The assumption $\mathcal{F}(Ue^{i\phi}) = \mathcal{F}(U)$ for any ϕ implies that $L(\mathbb{I}) = 0$. Indeed, then

$$0 = \frac{\partial \mathcal{F}(U_T^f e^{i\phi})}{\partial \phi} \bigg|_{\phi=0} = i \text{Tr} \left(\frac{\delta \mathcal{F}}{\delta U} U \bigg|_{U=U_T^f} \right) = -L(\mathbb{I})$$

If $f(t)$ is a critical control, then also $L(V_t) = 0$ and the Lemma implies $L \equiv 0$. Taking $A = X^\dagger$, we get $L(X^\dagger) = \text{Tr}(XX^\dagger) = 0$ and therefore $X = 0$. Since U_T^f is unitary, that implies $\delta \mathcal{F} / \delta U = 0$, i.e. f is an extrema of the functional $\mathcal{F}(f)$ if and only if $U = U_T^f$ is an extrema of the function \mathcal{F} . Hence if the only extrema of \mathcal{F} are global maxima and global minima, then the same is true for $\mathcal{F}(f)$ apart possibly of the exceptional control $f = f_0$. The control $f(t) = f_0$ requires a separate analysis and is shown to be not a trap in Lemma 2. This completes the proof.

Remark 2 *The statement of Theorem 1 is non-trivial and is a special property of two-level systems. In general, the trap-free property of $\mathcal{F}(U)$ might not imply the trap-free property of $\mathcal{F}(f)$ as was shown for various n -level systems with $n \geq 3$ [25, 29].*

Remark 3 *The statement of Lemma 1 means that the map $f \rightarrow U_T^f$ has the maximal rank at each point U_T^f of the unitary group $\text{SU}(2)$ because the gradient $\nabla_f U_T^f$ is surjective on the tangent bundle of $\text{SU}(2)$. In this case, as follows from [32, Theorem 1], the critical points of the kinematic landscape are in bijective correspondence with the critical points of the dynamic landscape. Thus, if the kinematic landscape has not only global maxima and global minima but also saddle points, then the statement of Theorem 1 about the absence of traps remains valid.*

While this theorem can be used to prove the absence of traps for objectives \mathcal{F}_O and \mathcal{F}_W , below we treat these important cases independently.

Theorem 2 *Let $\text{Tr}V = 0$ and $T \geq \pi/(\|H_0 - \frac{1}{2}\text{Tr}H_0 + f_0V\|)$. Then the only extrema of $\mathcal{F}_O(f)$ (hence also of $\mathcal{F}_{i \rightarrow f}(f)$ as well) are global maxima and global minima.*

Proof. Consider first the case $f \neq f_0$. The gradient of the objective \mathcal{F}_O is $\nabla \mathcal{F}_O(t) = L_O(V_t)$, where the map $L_O : \mathcal{M}_2 \rightarrow \mathbb{R}$ is defined by

$$L_O(A) = -i\text{Tr}([\rho_0, O_T]A)$$

with $O_T = U_T^{f\dagger} O U_T^f$. At any critical control, $\nabla \mathcal{F}_O = 0$ and hence $L_O(V_t) = 0$. Clearly, $L_O(\mathbb{I}) = 0$. Then the Lemma implies that $L_O \equiv 0$. In this proof, we denote by $|f\rangle$ vector such that $O_T = |f\rangle\langle f|$, and denote by $|f_\perp\rangle$ vector orthogonal to $|f\rangle$. Now take the operators $A = |f_\perp\rangle\langle f| + |f\rangle\langle f_\perp|$ and $A' = i(|f_\perp\rangle\langle f| - |f\rangle\langle f_\perp|)$. The equalities $L_O(A) = 0$ and $L_O(A') = 0$ imply $\text{Im}\langle f_\perp|\rho_0|f\rangle = 0$ and $\text{Re}\langle f_\perp|\rho_0|f\rangle = 0$, respectively. Hence $\langle f_\perp|\rho_0|f\rangle = 0$ and therefore $|f\rangle$ is an eigenstate of ρ_0 . Its only possible eigenvalues are ω_0 and ω_1 that correspond to the global minimum ($\mathcal{F}_O^{\min} = \langle f|\rho_0|f\rangle = \omega_0$) and the global maximum ($\mathcal{F}_O^{\max} = \langle f|\rho_0|f\rangle = \omega_1$) of the objective, respectively. These are the only allowed critical points except of $f(t) \equiv f_0$. The proof for the exceptional case f_0 follows from Lemma 2.

Theorem 3 *Let $\text{Tr}V = 0$ and $T \geq \pi/(\|H_0 - \frac{1}{2}\text{Tr}H_0 + f_0V\|)$. Then the only extrema of $\mathcal{F}_W(f)$ are global maxima and global minima.*

Proof. First we consider the case $f \neq f_0$. The gradient of the objective \mathcal{F}_W has the form $\nabla \mathcal{F}_W(t) = L_W(V_t)$, where the map $L_W : \mathcal{M}_2 \rightarrow \mathbb{R}$ is defined by

$$L_W(A) = \frac{1}{2} [\Im \text{Tr}Y \cdot \Re \text{Tr}(YA) - \Re \text{Tr}Y \cdot \Im \text{Tr}(YA)]$$

Here $Y = W^\dagger U_T^f$ (Y is unitary). Clearly, $L_W(\mathbb{I}) = 0$. At any critical control $\nabla \mathcal{F}_W = 0$ and hence Lemma 1 implies that $L_W \equiv 0$. We consider the operators $A = Y + Y^\dagger$ and $A' = i(Y - Y^\dagger)$ and denote $\text{Tr}Y = y_{\Re} + iy_{\Im}$ and $\text{Tr}Y^2 = z_{\Re} + iz_{\Im}$, where y_{\Re} and y_{\Im} are real and imaginary parts of $\text{Tr}Y$, z_{\Re} and z_{\Im} are real and imaginary parts of $\text{Tr}Y^2$. The equalities $L_W(A) = L_W(A') = 0$ become

$$\begin{aligned} z_{\Im}y_{\Re} - (z_{\Re} + 2)y_{\Im} &= 0 \\ (z_{\Re} - 2)y_{\Re} + z_{\Im}y_{\Im} &= 0 \end{aligned}$$

The solution $y_{\Re} = y_{\Im} = 0$ corresponds to the global minimum of the objective ($\mathcal{F}_W^{\min} = 0$). The compatibility of the system for other solutions requires $z_{\Im}^2 + z_{\Re}^2 \equiv |\text{Tr}Y^2|^2 = 4$. Since $\mathcal{F} = (1/4)|\text{Tr}Y|^2 = (1/4)|\text{Tr}Y^2|^2$, that implies that these solutions correspond to the global maximum $\mathcal{F}_W^{\max} = 1$ and no other solutions exist. The proof for the exceptional case f_0 follows from Lemma 2.

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